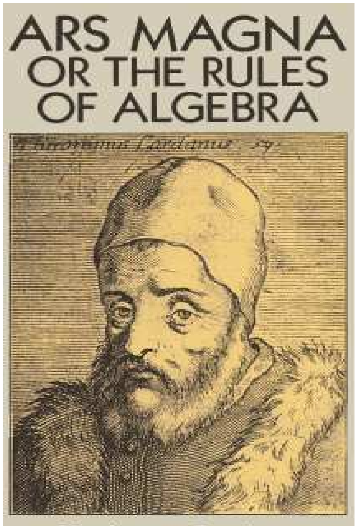
IMAGINARY NUMBER

In our college mathematics probably we all have learned about the imaginary number i. We were simply introduced i as a number.
Is it a number?
What kind of number is i ?
Where does the name *imaginary* come from?
Is it really an imaginary number? If yes then how did this number get involved into the real mathematics?
Can *imaginary number* do the same thing or different thing as *real number* do or more?
British mathematical physicist Roger Penrose calls i magic number. What is magical about i?
Let us explore this in this short article.

In algebra we must be careful with square-roots, especially when using negative numbers, because, by definition, the square of a positive number is always positive, and the square of a negative number is again positive (and the square of 0 is just 0 again, so that is hardly of use to us here). It seems impossible that we can find a number whose square is actually negative.. This, however, did not prevent mathematicians from finding a way around such an inconvenience—for in the 16th century, Girolamo Cardano [1501–1576] and Rafael Bombelli [1526–1572] were trying to make sense of this irreducible object.


Rafael Bombelli



Girolamo Cardano

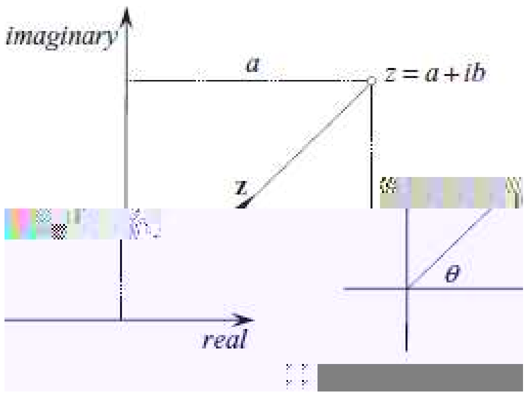
It was Bombelli who demonstrated that $\sqrt{-1}$ could be embraced by elementary algebra so long as such objects were left undisturbed. Eventually, in 1777 the brilliant German mathematician, Leonhard Euler [1707–1783], introduced the symbol i to stand in for $\sqrt{-1}$ which permitted expressions such as $2+\sqrt{-3}$ to be expressed as $2+i\sqrt{3}$. What kind of number it is? Is it real or imaginary or something else? The role of the '$+$' sign in this context is rather strange as it is impossible to coalesce the two terms into a single value. Isn't it like someone asking you *how many apple do you have after adding 5 mangoes to 6 oranges?* Actually the answer is *yes*. Mathematicians call it *Complex number*.

John Stillwell wrote in his book "*This resolution of the paradox of* $\sqrt{-1}$ *was so powerful, unexpected, and beautiful that only the word "miracle" seems adequate to describe it*".
The birth of complex number was originated with the solution of cubic equation.

It is also rather unfortunate that i or $\sqrt{-1}$ are referred to as *imaginary* quantities, as there is nothing imaginary about them — i just has the useful feature that $i^2 = -1$. We have René Descartes to blame for this.

The English mathematician, John Wallis [1616–1703], published his *De algebra tractatus* in 1685 in which he described a method of representing complex numbers as points on a plane. Unfortunately, his description was rather obtuse and failed to influence mathematicians of the day. Paul Nahin's book *An Imaginary Tale: The Story of* $\sqrt{-1}$ records that the Norwegian surveyor, Caspar Wessel [1745–1818], was another person to discover the geometric interpretation of complex numbers. His paper entitled "*On the Analytic Representation of Direction: An Attempt*" was announced in 1797 and in 1799 was published in a local Danish journal with a small international circulation, which ensured that his idea remained hidden for almost 100 years! Wessel's paper was discovered in 1895 by an antiquarian and its importance recognized by the Danish mathematician, Christian Juel [1855–1935], but it was too late—the Swiss-born writer, Jean-Robert Argand [1768–1822], had also thought of the same idea in 1806, and it is Argand's name that is associated with the *complex plane* rather than Wessel's.


John Wallis

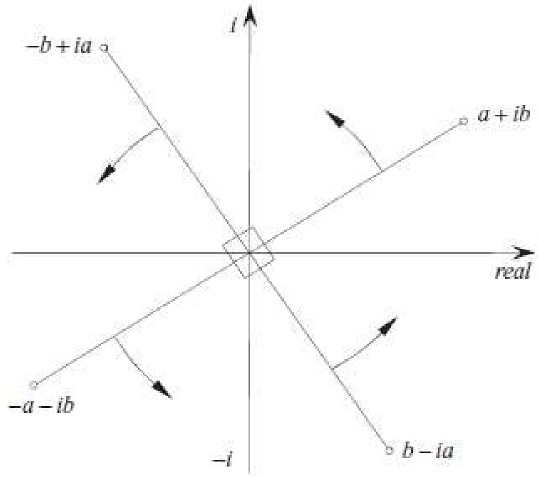



Jean Robert Argand

The complex plane, or *Argand diagram*, uses two orthogonal axes to locate a complex number: the horizontal axis is the real axis whilst the vertical axis is the imaginary axis.

i as a rotational operator (rotor)

Once Wessel had discovered a geometric interpretation for complex numbers, the rotational qualities of i became apparent. To illustrate this, let's trace the path of a complex number as it is repeatedly multiplied by i. We start with $a + ib$. Multiplying this by i we obtain $-b + ia$. Multiplying again we obtain $-a - ib$. Multiplying again we obtain $b - ia$, and finally $a + ib$. After four multiplications we return to the original complex number. When these intermediate numbers are located on an Argand diagram we see that each multiplication by i results in an anticlockwise rotation of 90°.



Multiplying a complex number by $-i$ performs a clockwise rotation. For example

$$-i(a+ib)=-b-ia$$

which is confirmed by the adjacent figure.

Unifying e, i , $\sin$ and $\cos$

The following formula, known as Euler's formula, discovered by German mathematician and physicist Leonhard Euler (1707-1783),

$$e^{ix}=\cos x+i\sin x$$

For $x=\frac{\pi}{2}$ it becomes,

$$e^{i\frac{\pi}{2}}=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)=i$$

and for $x=\pi$

$$e^{i\pi}=\cos\pi+i\sin\pi=-1$$

After arranging,

$$e^{i\pi}+1=0$$

As Paul J. Nahin wrote about this equation in his book *Dr. Euler's fabulous formula*, "I think $e^{i\pi}+1=0$ is beautiful because it is true even in the face of *enormous* potential constraint. The equality is precise; the left-hand side is not "almost" or "pretty near" or "just about" zero, but *exactly* zero. That *five* numbers, each with vastly different origins, and each with roles in mathematics that cannot be exaggerated, should be connected by such a simple relationship, is just stunning. *It is beautiful*."

Now what is really strange about i is that it can even be raised to itself: i^i . To discover its value,
we rearrange Eq. (7.8):

$$e^{i\frac{\pi}{2}}=-1$$

$$e^{i\frac{\pi}{2}}=i$$

Now we introduce i^i :

$$i^i=\left(e^{i\frac{\pi}{2}}\right)^i=e^{i^2\frac{\pi}{2}}=e^{-\frac{\pi}{2}}$$

which is a real value!

To explore more, see

Vince, J. *Geometry for Computer Graphics Formulae, Examples and Proofs*, Springer, 2005.
Vince, J. *Geometric Algebra for Computer Graphics*, Springer, 2008.
Vince, J. *Rotation Transforms for Computer Graphics*, Springer, 2011.
Vince, J. *Quaternions for Computer Graphics*, Springer, 2011.
Nahin, P. *An Imaginary Tale The Story of* $\sqrt{-1}$, Princeton University Press, 1998, with a new preface and appendixes by the author, 2007.
No other book can provide you much information than this book. So read it.
Nahin, P. *Dr. Euler's Fabulous Formula,* Princeton University Press, 2006.
Penrose, R. *The Road to Reality A Complete Guide to the Laws of the Universe*, Jonathan Cape, 2004. This is one of the best-selling books on natural science. As the title indicates it is truly a complete guide.
Stillwell, J. *Mathematics and Its History,* Undergraduate Texts in Mathematics, 3rd Ed. Springer 2010.