Amicable Numbers

A pair of numbers is amicable if each is the sum of the proper divisors of the other. The smallest pair is 220 and 284. The proper divisors of sum to,

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

and similarly,

$$1 + 2 + 4 + 71 + 142 = 220.$$

According to the philosopher Iamblichus (c. AD 250–330), the followers of Pythagoras “call certain numbers amicable numbers, adopting virtues and social qualities to numbers, such as 284 and 220; for the parts of each have the power to generate the other,” and Pythagoras described a friend as “one who is the other I, such as are 220 and 284.”

The first few amicable pairs are: (220, 284), (1184, 1210), (2620, 2924), (5020, 5564), (6232, 6368)

In the Bible (Genesis 32:14), Jacob gives 220 goats (200 female and 20 male) to Esau on their reunion. There are other biblical references at Ezra 8:20 and 1 Chronicles 15:6, while 284 occurs in Nehemiah 11:18. These references are all to the tribe of Levi, whose name derives from the wish of Levi’s mother to be amicably related to his father. They were also used in magic and astrology. Ibn Khaldun (1332–1406) wrote that “the art of talismans has also made us recognize the marvelous virtues of amicable (or sympathetic) numbers. These numbers are 220 and 284. . . . Persons who occupy themselves with talismans assure that these numbers have a particular influence in establishing union and friendship between individuals.” (Ore 1948, 97)

Thabit ibn Qurra (c. AD 850) in his *Book on the Determination of Amicable Numbers* noted that if you choose n so that each of the expressions $a = 3 \cdot 2^n - 1$, $b = 3 \cdot 2^{n-1} - 1$, and $c = 9 \cdot 2^{2n-1} - 1$ is prime, then $2^n ab$ and $2^n c$ are amicable numbers. Unfortunately, it isn’t easy to make them all prime at once, and in fact it only works for $n = 2, 4, 7$ and no other n less than 20,000.

A second pair, 17,296 and 18,416, was discovered by Ibn al-Banna (1256–1321) and rediscovered by Fermat in 1636. Descartes found the third pair, 9,363,584 and 9,437,056, which is the case $n = 7$ in Thabit’s formulae. Euler then discovered no less than sixty-two more examples, without following Thabit’s rule.

Paganini’s amicable pair, 1184 and 1210, is named after Nicolo Paganini, who discovered them in 1866 when he was a sixteen-year old schoolboy. They had previously been missed by Fermat, Descartes, Euler, and others.

More than 7,500 amicable pairs have been found, using computers, including all pairs up to 10^{14}. Is there an infinite number of amicable pairs? It is generally believed so, partly because Herman te Riele has a method of constructing “daughter” pairs from some “mother” pairs. Te Riele has also published all of the 1,427 amicable pairs less than 10^{10}

The following table summarizes the largest known amicable pairs discovered in recent years. The largest of these is obtained by defining

$$a = 2 \cdot 5 \cdot 11$$
$$S = 37 \cdot 173 \cdot 409 \cdot 461 \cdot 2136109 \cdot 2578171801921099 \cdot 68340174428454377539$$

$$p = 925616938247297545037380170207625962997960453645121$$

$$q = 210958430218054117679018601985059107680988707437025081922673599999$$

$$q_1 = (p + q)p^{235} - 1$$

$$q_2 = (p - S)p^{235} - 1$$

then p, q, q_1 and q_2 are all primes, and the numbers

$$n_1 = a\,S\,p^{235}q_1$$

$$n_2 = a\,q\,p^{235}q_2$$

are an amicable pair, with each member having 24,073 decimal digits.

Amicable Curiosities

- There is no known amicable pair in which one number is a square.
- The numbers in amicable pairs end in 0 or 5 surprisingly often, for no known reason.
- Most amicable numbers have many different factors. Can a power of a prime, p^n, be one of an amicable pair? If so, then $p^n > 10^{1500}$ and $n > 1400$.
- It is not known whether there is a pair of coprime amicable numbers. If there is, the numbers must exceed 10^{25} and their product must have at least twenty-two distinct prime factors.

Excerpted from the book
David Wells *Prime Numbers The Most Mysterious in Math*, John Wiley and Sons Inc., 2005.

Weisstein, Eric W. "Amicable Pair." From *MathWorld*--A Wolfram Web Resource.
http://mathworld.wolfram.com/AmicablePair.html